1 Introduction

In 1995, Edward Witten showed in his famous paper [1] that all the five ten-dimensional string theories are related through a web of dualities. In particular, Type IIA string theory can be obtained by compactifying an eleven dimensional theory called M-theory on $S^1 \times R^{1,9}$. At low energies, this theory, which we know little about as of now, has an effective description given by 11-dimensional supergravity.

In the rest of this report, we will first review Type IIA supergravity theory and eleven-dimensional supergravity in Section II. In section III, we will explain how the compatification reduces 11-dimensional supergravity to Type IIA supergravity. In section IV, we will explain how the strings and branes in Type IIA string theory arise from compactification.

2 Low-energy Effective Actions

In the target space, the excitations of superstrings can be identified with different massless or massive states. The mass of massive states $M$ is proportional to the square root of the string tension $\sqrt{T}$ so when the string tension is large, the massive states become too heavy to be excited. In this limit, superstring theories can be described effectively by their corresponding supergravity theories with only massless excitations. The low energy effective description for M-theory and Type IIA string theory are 11-dimensional supergravity and Type IIA supergravity respectively.

11-dimensional supergravity contains a graviton $G_{MN}$, a gravitino $\Psi^i_M$ and a three-form gauge field $A_3$ with gauge symmetry $A_3 \rightarrow A_3 + d\Lambda$. The graviton is a symmetric traceless tensor in an irreducible representation of the little group $SO(9)$. The gravitino is a spin 3/2 majorana fermion with 32 spinor components; its spin index transforms in $Spin(9)$ and the
vector index transforms in SO(9). The bosonic action of the theory is given by,

\[ S = \frac{1}{16\pi G_{N11}} \left[ \int d^{11}x \sqrt{-G} \left( R - \frac{1}{2} |dA_3|^2 \right) - \frac{1}{6} \int A_3 \wedge dA_3 \wedge dA_3 \right] \] (1)

The absence of a two-form gauge field suggests that the fundamental objects in M-theory are not strings. Instead, they are extended objects called membranes! In D dimensions, a p-form couples to (p − 1)-branes electrically and (D − p − 3)-branes magnetically. Therefore, the fundamental objects in M-theory are M2-branes and M5-branes.

Type IIA supergravity lives in ten dimensions and consists of fields from four sectors. The four sectors come from the different boundary conditions of the holomorphic and the anti-holomorphic oscillators which can be either NS type or R type. We denote the sectors according to their boundary conditions: NS-NS, R-R, NS-R, R-NS. The NS-NS and R-R sector contains bosons while the NS-R and the R-NS sector contains fermions. The field content includes:

**Bosons**
- NS-NS sector: a graviton \( g_{\mu\nu} \), a two-form gauge field \( B_{\mu\nu} \) and a dilaton \( \Phi \)
- R-R sector: a one-form \( A_1 \) and a three-form \( A_3 \)

**Fermions**
- NS-R sector: a left-handed gravitino \( \psi_{\mu}^+ \) and a left-handed dilatino \( \chi_{\mu}^+ \)
- R-NS sector: a right-handed gravitino \( \psi_{\mu}^- \) and a right-handed dilatino \( \chi_{\mu}^- \)

Both gravitini and dilatini are Majorana-Weyl fermions with 16 spinor components. The gauge fields combine into four field strengths, \( H_3 = dB, F_2 = dA_1, F_4 = dA_3, \tilde{F}_4 = dA_3 + A_1 \wedge H_3 \). The bosonic action \( I = I_{NS} + I_R + I_{CS} \) can be split into three terms. The action of the NS-NS sector,

\[ I_{NS} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} e^{-2\Phi} \left[ R + 4(\nabla \Phi)^2 - \frac{1}{2} |H_3|^2 \right] \] (2)

The action of the R-R sector,

\[ I_R = -\frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} \left[ \frac{1}{2} |F_2|^2 + \frac{1}{2} |\tilde{F}_4|^2 \right] \] (3)

and a Chern-Simons term:

\[ I_{CS} = -\frac{1}{2\kappa^2} \int d^{10}x \frac{1}{2} B_2 \wedge F_4 \wedge F_4 \] (4)

The two-form \( B_{\mu\nu} \) in the NS-NS sector couples to NS5-branes and string worldsheets which are 1-branes called fundamental strings. Type IIA string theory also contains other excitations including D0-branes, D2-branes, D4-branes and D6-branes which couple to either one-form potential \( A_1 \) or the three-form potential \( A_3 \).
3 Compactification of M-Theory

We are now ready to show that Type IIA supergravity can be found from 11-dimensional supergravity by the technique of dimensional reduction. In compactification, all the modes in the Fourier expansion of the various fields along the compactified direction are kept in the lower-dimensional theory. However, in dimensional reduction, one dimension is taken to be a circle and only the zero modes are kept because non-zero modes have non-vanishing masses in lower-dimensions and are therefore beyond the massless sectors.

We compactify the eleventh dimension into a circle of radius $R$ and assume that this direction is translational invariant, therefore, the fields have no dependence on the eleventh dimension. We take $\mu, \nu, \rho, \lambda$ to take the values $0, 1, \ldots, 9$, and $M, N$ to take the values $0, 1, \ldots, 9, 11$ (we skip 10).

3.1 Field Redefinition

We first consider dimensional reduction of gravitino. The $\Gamma^{11}$ operator from the eleven-dimensional Dirac algebra acts as the chirality operator in ten dimensions. $\Gamma^{11}$ operator can be diagonalized into $\text{diag}(1_{16 \times 16}, -1_{16 \times 16})$. Then, the first 16 components and the second 16 components of the eleven-dimensional spinor becomes the left-handed and right-handed Weyl spinors in ten dimensions. Therefore, the first 10 space time components of the gravitino $\Psi^i_M$ reduce to two opposite chiral gravitini $\psi^{+i}_\mu = \Psi^i_{\mu}$, $\psi^{-i}_\mu = \Psi^{16+i}_\mu$. The eleventh space time component of the gravitino similarly splits into two opposite chiral dilatini $\chi^{+i} = \Psi^i_{11}$, $\chi^{-i} = \Psi^{16+11}_i$ in ten dimensions. The reduction of the fermionic sector is straightforward after the field redefinition.

The dimensional reduction proceeds in the following manner: we write in the bosonic sector, the graviton in eleven dimensions can be decomposed into a ten-dimensional graviton $g_{\mu\nu}$, a one-form potential $A_\mu$ and a scalar dilaton $\Phi$ in ten dimensions,

$$G_{MN} = e^{-2\Phi/3} \left[ g_{\mu\nu} + e^{2\Phi} A_\mu A_\nu e^{2\Phi} A_\mu \right]$$

(5)

The eleven-dimensional three-form $A^{(11)}_{MNP}$ reduces to a three-form $A_{\mu\nu\rho} = A^{(11)}_{\mu\nu\rho}$ and a two-form $B_{\mu\nu} = A^{(11)}_{\mu\nu11}$ in ten dimensions. The corresponding field strength also reduces to the fields strengths in ten dimensions $(F_4)_{\mu\nu\rho\lambda} = F^{(11)}_{\mu\nu\rho\lambda}$ and $(H_3)_{\mu\nu\rho} = F^{(11)}_{\mu\nu\rho11}$.

3.2 Action Reduction

Using Cartan formalism (refer to Appendix I), we can determine the dimensional reduction of field strength and Ricci scalar. Because of the flatness of the metric, it is convenient to do the calculation at tangent space. At the tangent space, the four-form field strength $F^{(11)} = dA^{(11)}_3$ in eleven dimensions reduces to a four-form field strength $F^{(11)}_{abcd} = e^{4\Phi/3} (F_4)_{abcd}$ and a three-form field strength $F^{(11)}_{abc11} = e^{\Phi/3} (H_3)_{abc}$ at ten dimensions. The eleven-dimensional Ricci
scalar reduces to the sum of the ten-dimensional Ricci scalar, the kinetic term of the dilaton and the Maxwell action of the one-form $A_\mu$: $R^{(11)} = e^{2\Phi/3} \left[ R^{(10)} + 4(\nabla \Phi)^2 + \frac{1}{2} e^{2\Phi} |F|^2 \right].$

$$S = \frac{1}{16\pi G_{N11}} \int d^{11}x \sqrt{-G} \left( R^{(11)} - \frac{1}{2} |dA_3^{(11)}|^2 \right) - \frac{1}{6} \frac{1}{16\pi G_{N11}} \int A_3^{(11)} \wedge dA_3^{(11)} \wedge dA_3^{(11)}$$

$$= \frac{2\pi R_{11}}{16\pi G_{N11}} \int d^{10}x \sqrt{-g} e^{-2\Phi} \left( R^{(10)} + 4(\nabla \Phi)^2 + \frac{1}{2} e^{2\Phi} |F|^2 \right) -$$

$$\frac{2\pi R_{11}}{16\pi G_{N11}} \int d^{10}x \sqrt{-g} \left( \frac{1}{2} |\tilde{F}_4|^2 + e^{-2\Phi} \frac{1}{2} |H_3|^2 \right) - \frac{1}{2} \frac{2\pi R_{11}}{16\pi G_{N11}} \int B_2 \wedge dA_3 \wedge dA_3$$

The action is the same as the Type IIA action after identifying $4\pi R_{11} \kappa^2 = 16\pi G_{N11}$.

### 3.3 Coupling Constant

The eleventh-eleventh component of the metric is $e^{2\Phi/3}$. So integrating along the eleventh direction, we notice that the radius of the compatified circle is proportional to $e^{2\Phi/3}$, i.e. $R_{11} \approx e^{2\Phi/3}$. Since the vacuum expectation value of the operator $e^\Phi$ is the Type IIA superstring coupling constant $g_s$, we have a relation between string coupling constant and the radius of the compactified circle $R_{11}$, $R_{11} \approx g_s^2$. Thus, the radius of the eleventh dimension $R_{11}$ grows when the coupling constant $g_s$ becomes strong and the circle is decompactified. Therefore, the strong coupling limit of a Type IIA string theory is equivalent to an eleven-dimensional M-theory in flat space time.

### 4 Branes

Depending on whether or not the branes wrap around the eleventh dimension, the $M_p$-branes in M-theory can reduce to either $Dp$-branes (not wrapping) or $D(p - 1)$-branes (wrapping). So the $M2$-branes and the $M5$-branes in M-theory naturally give rise to fundamental strings (1-branes), $D2$-branes, $D4$-branes and $NS5$-branes in Type IIA string theory.

One would notice at this point that the $D0$-branes and $D6$-branes are missing from this description. While a full explanation on how these can be recovered is beyond the scope of this report, a short answer is that the $D0$-branes can be interpreted as the Kaluza-Klein modes of the compactification and $D6$-branes are the magnetic dual of the modes which are equivalent to the tensor product of five-dimensional magnetic monopoles and six-dimensional space.

### 5 Appendix I. Cartan Formalism

We apply the Cartan formalism to action reductions. The Cartan formalism requires the basic ingredient: an invertible linear map $e : V \rightarrow TM$, between vector bundles over $M$ where
$T M$ is the tangent bundle of $M$. For local coordinates and a local frame $\partial_\mu$ of the tangent bundle, the map $e$:

$$e_a = e^\mu_a \partial_\mu$$ (7)

where $a = 1, \ldots, n$, and $n$ is the dimension of $V$. The matrix $e^\mu_a$ is called the vielbein satisfying $g^{\mu\nu} = e^\mu_a e^\nu_b \eta^{ab}$. In our case, where $n = 11$, the vielbein is called an elfbein (elf being German for eleven), we denote the elfbein by $E^A_M$, and the reduction takes the form (in terms of $E^A_M$):

$$E^A_M = \begin{bmatrix} e^{-\frac{2\phi}{\tau}} e^a \mu & 0 \\ e^{\frac{2\phi}{\tau}} A_\mu & e^{\frac{2\phi}{\tau}} \end{bmatrix}$$ (8)

where $e^a_\mu$ is the 10-dimensional ‘zehnbein’. We will also make use of the inverse elfbein:

$$E^M_A = \begin{bmatrix} e^{\frac{\phi}{\tau}} e^\mu_a & 0 \\ -e^{\frac{\phi}{\tau}} A_a & e^{-\frac{2\phi}{\tau}} \end{bmatrix}$$ (9)

The dimensional reduction formulae is much simpler when we move to tangent-space indices since the tangent-space metric is automatically diagonal. We first consider the dimensional reduction of field strength. There are two cases of $F_{ABCD}^{(11)} = E^M_A E^N_B E^P_C E^Q_D F_{MNPQ}^{(11)}$, one case being when all indices are ten-dimensional, and the other case where one is 11-dimensional:

$$F^{(11)}_{abcd} = E^M_A E^N_B E^P_C E^Q_D F_{MNPQ}^{(11)}$$

$$= E^\mu_a E^\nu_b E^\rho_c E^{\lambda}_d F_{\mu\nu\rho\lambda}^{(11)} + E^\mu_a E^\nu_b E^\rho_c E^{\lambda}_d F_{11\nu\rho\lambda}^{(11)} + E^\mu_a E^\nu_b E^\rho_c E^{\lambda}_d F_{\mu11\rho\lambda}^{(11)} + E^\mu_a E^\nu_b E^\rho_c E^{\lambda}_d F_{\mu\nu11\rho\lambda}^{(11)}$$

$$= e^{\frac{4\phi}{\tau}} e^a_\mu e^b_\nu e^c_\rho e^d_\lambda F_{\mu\nu\rho\lambda}^{(11)} - e^{\frac{4\phi}{\tau}} A_a e^b_\nu e^c_\rho e^d_\lambda F_{11\nu\rho\lambda}^{(11)} - e^{\frac{4\phi}{\tau}} A_a e^b_\nu e^c_\rho e^d_\lambda F_{\mu11\rho\lambda}^{(11)} - e^{\frac{4\phi}{\tau}} A_a e^b_\nu e^c_\rho e^d_\lambda F_{\mu\nu11\rho\lambda}^{(11)}$$

$$- e^{\frac{4\phi}{\tau}} (F_{abcd} + 4A_{[a} H_{bcd]}) = e^{\frac{4\phi}{\tau}} F_{abcd}$$ (10)

$$F^{(11)}_{abc1} = E^M_A E^N_B E^P_C E^Q_D F_{MNPQ}^{(11)}$$

$$= E^\mu_a E^\nu_b E^\rho_c E^{\lambda}_d F_{\mu\nu\rho\lambda}^{(11)} + E^\mu_a E^\nu_b E^\rho_c E^{\lambda}_d F_{11\nu\rho\lambda}^{(11)} + E^\mu_a E^\nu_b E^\rho_c E^{\lambda}_d F_{\mu11\rho\lambda}^{(11)} + E^\mu_a E^\nu_b E^\rho_c E^{\lambda}_d F_{\mu\nu11\rho\lambda}^{(11)}$$

$$= E^\mu_a E^\nu_b E^\rho_c E^{\lambda}_d F_{\mu\nu\rho\lambda}^{(11)} + E^\mu_a E^\nu_b E^\rho_c E^{\lambda}_d F_{11\nu\rho\lambda}^{(11)} + E^\mu_a E^\nu_b E^\rho_c E^{\lambda}_d F_{\mu11\rho\lambda}^{(11)} + E^\mu_a E^\nu_b E^\rho_c E^{\lambda}_d F_{\mu\nu11\rho\lambda}^{(11)}$$

$$= E^\mu_a E^\nu_b E^\rho_c E^{\lambda}_d F_{\mu\nu\rho\lambda}^{(11)} = e^{\frac{\phi}{\tau}} e^a_\mu e^b_\nu e^c_\rho e^d_\lambda H_{\mu\nu\rho\lambda} = e^{\frac{\phi}{\tau}} H_{abc}$$ (11)

since $E^{\lambda}_d = 0$. Thus, the dimensional reduction of the 11-dimensional field strength is a combination of a four-form $F_4 = dA_3 + A_1 \wedge H_3$ and a three-form field strength $H_3 = dB$. 

5
References
